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A non-singular one-loop wave function of the universe from a new eigenvalue asymptotics in quantum gravity

Giampiero Esposito,[a] Guglielmo Fucci,[b] Alexander Yu. Kamenshchik[c] and Klaus Kirsten[d]

[a] *INFN, Sezione di Napoli and*
Dipartimento di Scienze Fisiche Università Federico II
Complesso Universitario di Monte S. Angelo
Via Cintia, Edificio N', 80126 Napoli, Italy
[b] *Department of Physics, New Mexico Institute of Mining and Technology*
Leroy Place 801, Socorro, NM 87801, U.S.A.
[c] *Dipartimento di Fisica and INFN, Sezione di Bologna*
Via Irnerio 46, 40126 Bologna, Italy, and
L.D. Landau Institute for Theoretical Physics
Kosygin str. 2, 119334 Moscow, Russia
[d] *Department of Mathematics, Baylor University*
Waco, TX 76798, U.S.A.
E-mail: giampiero.esposito@na.infn.it, gfucci@nmt.edu, alexander.kamenshchik@bo.infn.it, Klaus_Kirsten@baylor.edu

Abstract: Recent work on euclidean quantum gravity on the four-ball has proved regularity at the origin of the generalized ζ-function built from eigenvalues for metric and ghost modes, when diffeomorphism-invariant boundary conditions are imposed in the de Donder gauge. The hardest part of the analysis involves one of the four sectors for scalar-type perturbations, the eigenvalues of which are obtained by squaring up roots of a linear combination of Bessel functions of integer adjacent orders, with a coefficient of linear combination depending on the unknown roots. This paper obtains, first, approximate analytic formulae for such roots for all values of the order of Bessel functions. For this purpose, both the descending series for Bessel functions and their uniform asymptotic expansion at large order are used. The resulting generalized ζ-function is also built, and another check of regularity at the origin is obtained. For the first time in the literature on quantum gravity on manifolds with boundary, a vanishing one-loop wave function of the Universe is found in the limit of small three-geometry, which suggests a quantum avoidance of the cosmological singularity driven by full diffeomorphism invariance of the boundary-value problem for one-loop quantum theory.

Keywords: Models of Quantum Gravity.

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Contents

1. Introduction

The subject of boundary effects in quantum field theory [1] has always received a careful consideration in the literature by virtue of very important physical and mathematical motivations, that can be summarized as follows.

(i) Boundary data play a crucial role in the functional-integral approach [2], in the quantum theory of the early universe [3], in supergravity [4] and even in string theory [5].

(ii) The way in which quantum fields react to the presence of boundaries is responsible for remarkable physical effects, e.g. the attractive Casimir force among perfectly conducting parallel plates, which can be viewed as arising from differences of zero-point energies of the quantized electromagnetic field [6].

(iii) The spectral geometry of a riemannian manifold [7] with boundary is a fascinating problem where many new results have been derived over the last few years [8, 9].

(iv) Boundary terms [10] in heat-kernel expansions [11] have become a major subject of investigation in quantum gravity [9], since they shed new light on one-loop conformal anomalies ([12, 13]) and one-loop divergences [14].

Within such a framework, recent work by the authors [15] has studied in detail the spectral asymptotics of euclidean quantum gravity on the four-ball, motivated by the long-standing problem of finding local and diff-invariant boundary conditions on metric perturbations that are compatible with self-adjointness and strong ellipticity (see the appendix) of the boundary-value problem relevant for one-loop quantum theory [16]. Interestingly, we have found that only one of the four eigenvalue conditions (i.e. the equations obeyed by the eigenvalues by virtue of boundary conditions) for scalar modes is responsible for lack of strong ellipticity that was expected, on general ground, from the work in [17]. Moreover, the spectral coefficients $K_a^{(j)}$ (see [15, section 5]) occurring in the uniform asymptotic expansion of the logarithmic derivative of such an eigenvalue condition:

$$F_B^{+}(n,x) \equiv J_n'(x) + \left(\frac{n}{x} - \frac{x}{2}\right) J_n(x) = 0\,, \qquad \forall n \geq 3\,, \tag{1.1}$$

ensure the validity, $\forall j = 1, \ldots, \infty$, of the peculiar spectral identity

$$\sum_{a=j}^{4j} \frac{\Gamma(a+1)}{\Gamma(a-j+1)} K_a^{(j)} = 0\,. \tag{1.2}$$

Equation (1.2) engenders, in turn, regularity at the origin of the generalized ζ-function built from the eigenvalues $E_i = x_i^2$, x_i being the roots of (1.1) (see below). Thus, a non-trivial example is found where the $\zeta(0)$ value remains meaningful even though strong ellipticity of the boundary-value problem is violated.

The aim of the present paper is to perform a numerical and analytical investigation of (1.1) which, besides being of intrinsic mathematical interest, may be of some help in understanding euclidean quantum gravity on the four-ball. Such a background is not an oversimplification, since simple supergravity with massless gravitinos, which already makes sense only on Ricci-flat four-manifolds [18] if the boundary is empty, is further restricted to flat euclidean four-manifolds if a local description of spin 3/2 in terms of two sets of potentials is exploited (see a detailed proof in [19]).

Sections 2 and 3 are devoted to numerical and analytical evaluation of roots of (1.1). Section 4 obtains the resulting asymptotic expansion of the generalized ζ-function, while concluding remarks, with emphasis on quantum cosmological implications, are presented in section 5.

2. Numerical evaluation of roots

If the Bessel function J_n is an even (resp. odd) function of x, its first derivative J_n' is an odd (resp. even) function of x. Thus, roots of (1.1) occur always in equal and opposite pairs $y_i = (x_i, -x_i)$, and the roots x_i are sufficient to build the generalized ζ-function from the eigenvalues $E_i = x_i^2$ [15]. Strictly, we should write $E_i = \frac{x_i^2}{q^2}$, where q is the three-sphere radius, S^3 being the boundary in our case, but we set $q = 1$ for simplicity. In [15] we have also studied the eigenvalue condition

$$F_B^{-}(n,x) \equiv J_n'(x) - \left(\frac{n}{x} + \frac{x}{2}\right) J_n(x) = 0\,, \qquad \forall n \geq 3\,, \tag{2.1}$$

and hereafter we present a table of numerical roots in the two cases. The vanishing root does not contribute to the generalized ζ-function, and hence it is not included.

As one can see, there exist a countable infinity of roots $y_i(F_B^{\pm})$ for each value of $n \geq 3$; they are very close in that

$$\lim_{i \to \infty} \left| y_{i+1}(F_B^{+}) - y_i(F_B^{-}) \right| = 0\,. \tag{2.2}$$

We also plot, to provide an example, the functions $F_B^{+}(5,x)$ and $F_B^{-}(5,x)$ (for larger values of n, the plots of $F_B^{\pm}(n,x)$ are harder to visualize, since the first root moves away from the origin).

F_B^+	1	2	3	4	5
n=3	3.05424	6.70613	9.96947	13.1704	16.3475
n=5	4.10467	9.0174	12.5069	15.8305	19.0872
n=20	8.72974	25.5005	30.0311	34.0494	37.8272
n=50	14.0029	57.153	62.8403	67.7276	72.2186
n=100	19.9008	108.855	115.757	121.592	126.887

Table 1: The first 5 roots of F_B^+ if $n = 3, 5, 20, 50, 100$.

F_B^-	1	2	3	4	5
n=3	6.63569	9.94638	13.1603	16.3422	19.5097
n=5	8.96599	12.4873	15.8207	19.0815	22.3053
n=20	25.4909	30.0253	34.0454	37.8242	41.4602
n=50	57.1508	62.8387	67.7263	72.2175	76.4628
n=100	108.854	115.756	121.592	126.886	131.839

Table 2: The first 5 roots of F_B^- if $n = 3, 5, 20, 50, 100$.

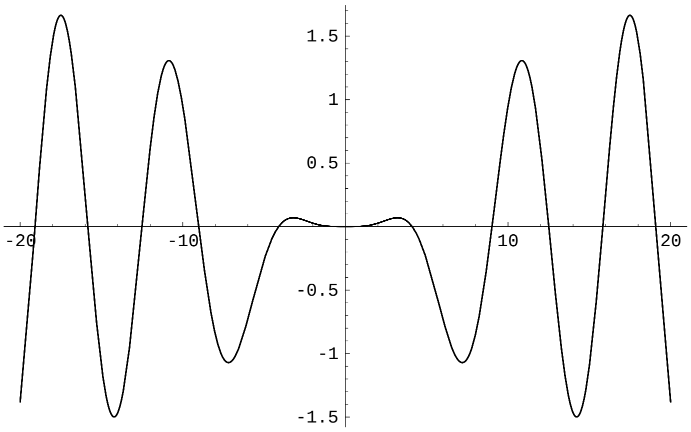

Figure 1: The function $F_B^+(5, x)$.

3. Approximate analytic formulae for the roots

In the course of investigating the zeros of $J_n(x)$, or $J_n'(x)$ or suitable combinations of the two, the starting point, following the seminal work by McMahon [20], is the descending-series formula of Hankel, according to which, for all n,

$$J_n(x) = \sqrt{\frac{2}{\pi x}}\left[\varphi_n(x)\cos\left(x - \frac{\pi}{4} - n\frac{\pi}{2}\right) + \psi_n(x)\sin\left(x - \frac{\pi}{4} - n\frac{\pi}{2}\right)\right], \tag{3.1}$$

where

$$\varphi_n(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!(2x)^{2k}} \frac{\Gamma\left(n + 2k + \frac{1}{2}\right)}{\Gamma\left(n - 2k + \frac{1}{2}\right)}, \tag{3.2}$$

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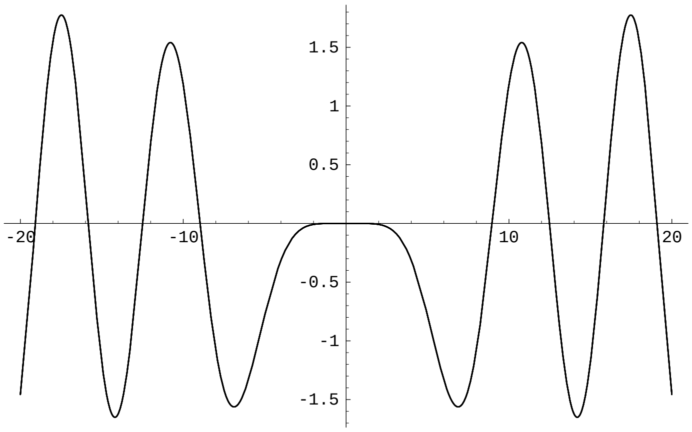

Figure 2: The function $F_B^-(5, x)$.

$$\psi_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!(2x)^{2k+1}} \frac{\Gamma\left(n+2k+\frac{3}{2}\right)}{\Gamma\left(n-2k-\frac{1}{2}\right)}. \tag{3.3}$$

We now exploit the identity

$$J_n'(x) + \frac{n}{x} J_n(x) = J_{n-1}(x) \tag{3.4}$$

to re-express the eigenvalue condition (1.1) in the form

$$J_n(x) - \frac{2}{x} J_{n-1}(x) = 0\,. \tag{3.5}$$

As a second step, we define

$$\theta_n(x) \equiv x - \frac{\pi}{4} - n\frac{\pi}{2}, \tag{3.6}$$

and rely upon eqs. (3.1)–(3.3) to further re-express the eigenvalue condition (3.5) in the form (hereafter $m(n) \equiv 4n^2$ as in [20])

$$\begin{aligned} \tan\theta_n(x) &= \frac{\frac{2}{x}\psi_{n-1}(x) - \varphi_n(x)}{\psi_n(x) + \frac{2}{x}\varphi_{n-1}(x)} \\ &= \frac{8x}{(m-17)} + \sum_{r=1}^{3} \frac{A_r(m(n))}{x^{2r-1}} + \mathrm{O}(x^{-7}), \end{aligned} \tag{3.7}$$

where the term linear in x distinguishes $\tan\theta_n$ from the purely Bessel case (in which $J_n(x) = 0$ and $\tan\theta_n$ only has inverse odd powers of x), and we find

$$A_r(m) = (m-17)^{-r-1} \sum_{k=0}^{6r} \alpha_{r,k} m^{k/2}\,, \tag{3.8}$$

where the only vanishing coefficients up to x^{-5} in (3.7) are

$$\alpha_{1,3}, \alpha_{1,5}, \alpha_{3,9}, \alpha_{3,11}, \alpha_{5,15}, \alpha_{5,17}\,.$$

The first term on the second line of (3.7) is responsible for the new features with respect to the analysis of [20], where the roots of $J_n(x)$ *for all* n were evaluated approximately for the first time. For that problem, the author exploited the fact that, for an equation of the form

$$x = \beta + \frac{P}{x} + \frac{Q}{x^3} + \frac{R}{x^5} + \mathrm{O}(x^{-7}), \tag{3.9}$$

the method of iterated approximations gives

$$x = \beta \left[1 + \frac{P}{\beta^2} + \frac{(Q - P^2)}{\beta^4} + \frac{(R - 4PQ + 2P^3)}{\beta^6} + \mathrm{O}(\beta^{-8}) \right]. \tag{3.10}$$

In our case we have to find, from (3.7), approximate solutions of the equation (s being any integer, here introduced to take into account periodicity of the tan function)

$$\theta_n(x) - s\pi = \arctan \left(\frac{8x}{(m-17)} + \sum_{r=1}^{3} \frac{A_r(m(n))}{x^{2r-1}} + \mathrm{O}(x^{-7}) \right). \tag{3.11}$$

On defining the variable $w \equiv x^{-1}$, the right-hand side of (3.11) can be Taylor expanded about $w = 0$, and this leads to the equation

$$x = \beta(s,n) + \sum_{r=1}^{3} \frac{B_r(m(n))}{x^{2r-1}} + \mathrm{O}(x^{-7}), \tag{3.12}$$

which is of the form (3.9), with (here $s \geq 0$)

$$\beta(s,n) \equiv \pi \left(s + \frac{n}{2} + \frac{3}{4} \right), \tag{3.13}$$

$$B_1(m) = P = -\frac{(m-17)}{8}, \tag{3.14}$$

$$B_2(m) = Q = -\frac{1721}{384} + 2m^{1/2} - \frac{35}{192}m - \frac{1}{384}m^2, \tag{3.15}$$

$$B_3(m) = R = \frac{79201}{5120} - \frac{47}{4}m^{1/2} + \frac{14973}{5120}m - \frac{1}{4}m^{3/2} + \frac{7m^2}{1024} - \frac{m^3}{5120}. \tag{3.16}$$

Our approximate roots read therefore as

$$x(s,n) \sim \beta(s,n) \left[1 + \frac{\gamma_1}{\beta^2(s,n)} + \frac{\gamma_2}{\beta^4(s,n)} + \frac{\gamma_3}{\beta^6(s,n)} + \mathrm{O}(\beta^{-8}) \right], \tag{3.17}$$

where, from (3.10),

$$\gamma_1 = B_1 = -\frac{(m-17)}{8}, \tag{3.18}$$

$$\gamma_2 = B_2 - B_1^2 = -\frac{3455}{384} + 2m^{1/2} + \frac{67}{192}m - \frac{7}{384}m^2, \tag{3.19}$$

$$\begin{aligned} \gamma_3 &= B_3 - 4B_1B_2 + 2B_1^3 \\ &= \frac{1117523}{15360} - \frac{115}{4}m^{1/2} - \frac{5907}{5120}m + \frac{3}{4}m^{3/2} + \frac{421}{3072}m^2 - \frac{83}{15360}m^3, \end{aligned} \tag{3.20}$$

F_B^+	1	2	3	4	5
n=3	3.05424	6.70613	9.96947	13.1704	16.3475
n=5	4.10467	9.0174	12.5069	15.8305	19.0872
x(s,3)	3.13152	6.70672	9.96951	13.1704	16.3475
x(s,5)	5.00988	9.03333	12.5089	15.831	19.0873

Table 3: Numerical roots of (3.5).

which provides a good approximation at large x and low values of n, as is further discussed below. Unlike the Dirichlet or Neumann problems, our γ_p are, in general, non-analytic functions of m.

Note that large x and small n actually also means large s; this is the only way eq. (3.17) provides a systematic asymptotic expansion with smaller and smaller correction terms. The roots in table 3 are well approximated by their analytic form (3.17) at large x and low n. For $n = 3, 5$, the first two lines display all numerical roots of (3.5) from the first to the fifth, while the third and fourth line show their approximation $x(s, 3)$ and $x(s, 5)$ from (3.17), with $s = 0, 1, 2, 3, 4$ (recall that the first root is obtained for $s = 0$). Remarkably, eqs. (3.17)–(3.20) approximate very well all roots starting from $x(1, n)$, while $x(0, n)$ is fairly well approximated only for the lowest value of n, i.e. $n = 3$.

The expansion (3.17) of the roots of (3.5), which holds for small n and large x, should be replaced by a different expansion at large n. For this purpose, one may start from the uniform asymptotic expansion of $J_n(x)$ and $J_n'(x)$ [21]. In terms of x the desired formulae read as (our polynomials u_k and v_k below can be found, for example, in [15, appendix B])

$$J_n(x) \sim \frac{e^{n\tilde{\eta}(n,x)}}{\sqrt{2\pi n}\left(1-\frac{x^2}{n^2}\right)^{1/4}} \left\{1+\sum_{k=1}^{\infty}\frac{u_k(t(n,x))}{n^k}\right\}, \tag{3.21}$$

and

$$J_n'(x) \sim \sqrt{\frac{n}{2\pi}}\,\frac{1}{x}\left(1-\frac{x^2}{n^2}\right)^{1/4} e^{n\tilde{\eta}(n,x)} \left\{1+\sum_{k=1}^{\infty}\frac{v_k(t(n,x))}{n^k}\right\}, \tag{3.22}$$

where

$$\tilde{\eta}(n,x) \equiv \frac{1}{n}\sqrt{n^2-x^2} - \log\left[\frac{n}{x}+\sqrt{\frac{n^2}{x^2}-1}\right], \tag{3.23}$$

and

$$t(n,x) \equiv \frac{1}{\sqrt{1-\frac{x^2}{n^2}}}\,. \tag{3.24}$$

On inserting (3.21) and (3.22) in (1.1), we obtain

$$F_B^+ \sim \frac{e^{n\tilde{\eta}}}{\sqrt{2\pi n}}\frac{(1+t)}{\sqrt{t}}\left(1-t^{-2}\right)^{-1/2}\left[1+\frac{(1-t)}{2t}(n+u_1)+\sum_{k=1}^{\infty}\frac{g_k(t)}{n^k}\right], \tag{3.25}$$

where

$$g_k(t) \equiv \frac{v_k(t)+tu_k(t)}{(1+t)} + \frac{(1-t)}{2t}u_{k+1}(t)\,. \tag{3.26}$$

F_B^+	$n=20$	$n=50$	$n=70$	$n=100$
$x(n)$	8.72974	14.0029	16.6152	19.9008
$y_3(n,x)$	8.9376	14.0029	16.6149	19.9007

Table 4: Exact vs. approximate roots of (3.5) at large n, with $x < n$.

It is very important, at this stage, to note explicitly that the expansion of our equation written above only holds for $x < n$. This can be understood on considering the functional form of (3.25) where, from (3.21) and (3.22), the variable x satisfies the inequality $x < n$. For this reason we can only find approximate solutions at large n of (3.5) when x is smaller than n. Looking at the table of numerical roots of our equation one can clearly note that, for large n, there is always only one root satisfying the condition $x < n$. Then the expansion here described can be used for approximating such roots.

For this purpose, let us define the following function (cf. (3.25)):

$$y_\lambda(n,x) \equiv 1 + \frac{(1-t)}{2t}(n+u_1) + \sum_{k=1}^{\lambda} \frac{g_k(t)}{n^k}\,. \tag{3.27}$$

One can obtain a good approximation of the roots by considering only the first three terms of the series which appears in the last equation. In this way one can get approximate values for the roots from the zeros of $y_\lambda(n,x)$ which are summarized, for a few values of n, in Table 4. An approximate formula for the roots $x(n)$ is obtained by setting $y_\lambda(n,x) = 0$. To leading order we find that such an equation reduces to

$$\frac{\sqrt{n^2-x^2}}{x} + \frac{n}{x} - \frac{x}{2} = 0\,,$$

which is solved by $x = \pm 2\sqrt{n}\sqrt{1-\frac{1}{n}}$. We therefore write $x = 2\sqrt{n} + c$, with c a constant. If we plug it into eq. (3.27) and expand for large values of n, and choose c so that the leading order vanishes, we find $c = 0$. Then we continue with $x \sim 2\sqrt{n} + 0 + \frac{c}{\sqrt{n}}$ and proceed in the same way to find $c = -1$. In this way we obtain, eventually,

$$x \sim \pm\left[2\sqrt{n} - \frac{1}{\sqrt{n}} + \frac{3}{4}n^{-3/2} + \frac{11}{8}n^{-5/2} - \frac{157}{64}n^{-7/2} + \mathrm{O}(n^{-9/2})\right]. \tag{3.28}$$

By looking at the first two terms of this expansion we discover that a better expansion parameter is actually $(n-1)$, and hence we obtain

$$x \sim \pm 2\sqrt{n-1}\left[1 + \frac{1}{2}(n-1)^{-2} - \frac{17}{8}(n-1)^{-4} + \mathrm{O}\left((n-1)^{-6}\right)\right]. \tag{3.29}$$

The fact that only every second term is non-vanishing in (3.29) suggests this is a good expansion parameter. It also explains why it approximates better than the expansion in terms of n. As the leading two terms show, an expansion of x in terms of $n-\delta$ for some $\delta \in]0,1[$ is not possible, because for large values of n one cannot satisfy the equation $y_\lambda(n,x) = 0$ from eq. (3.27).

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Oscillating asymptotic expansions of the roots as $x > n$ can also be obtained, but they are not written down for brevity. The leading term (3.13) in the expansion (3.12) of the roots can be used to reproduce two of the three contributions to $\zeta(0)$ found in [15]. This is reassuring, but there is no exact formula for $x(s,n)$ for all s, n, and hence it remains unclear how to evaluate $\zeta(0)$ without resorting to contour integration (see below).

4. Generalized ζ-functions from contour integrals

Tables 1 and 2 jointly with the limit in (2.2) tell us that, for any fixed n, the root $x(i,n,+)$ of F_B^+ is very close to and slightly larger than the root $x(i-1,n,-)$ of F_B^-. Thus, a $\rho(i,n)$ positive and much smaller than 1 exists such that, for the eigenvalues obtained by squaring up the roots, one can write

$$E(i,n,+) = E(1,n,+)\delta_{i,1} + E(i-1,n,-)(1+\rho(i,n))(1-\delta_{i,1}) \,, \tag{4.1}$$

for all $n \geq 3$ and for all $i \geq 1$, where i labels here the countable infinity of roots for any given n and hence starts from 1.

This suggests looking for a link among the generalized ζ-functions $\zeta_B^+(s)$ and $\zeta_B^-(s)$ (hereafter s is the independent variable in the generalized ζ-functions). Following the methods developed in [22, 23], our starting point is the integral representation in [15], i.e. (hereafter $\beta_+ \equiv n, \beta_- \equiv n+2$)

$$\zeta_B^{\pm}(s) \equiv \frac{(\sin \pi s)}{\pi} \sum_{n=3}^{\infty} n^{-(2s-2)}$$
$$\int_0^{\infty} dz \, z^{-2s} \frac{\partial}{\partial z} \log \left[z^{-\beta_\pm(n)} \left(z n I_n'(zn) + \left(\frac{z^2 n^2}{2} \pm n \right) I_n(zn) \right) \right] . \tag{4.2}$$

We exploit the uniform asymptotic expansion of modified Bessel functions and their first derivatives to find (hereafter $\tau \equiv (1+z^2)^{-1/2}$)

$$z n I_n'(zn) + \left(\frac{z^2 n^2}{2} \pm n \right) I_n(zn) \sim \frac{n^2}{2\sqrt{2\pi n}} \frac{e^{n\eta}}{\sqrt{\tau}} \left(\frac{1}{\tau} - \tau \right) \left(1 + \sum_{k=1}^{\infty} \frac{r_{k,\pm}(\tau)}{n^k} \right) , \tag{4.3}$$

where we have (bearing in mind that $u_0 = v_0 = 1$)

$$r_{k,\pm}(\tau) \equiv u_k(\tau) + \frac{2\tau}{(1-\tau^2)} \left((v_{k-1}(\tau) \pm \tau u_{k-1}(\tau) \right) , \tag{4.4}$$

for all $k \geq 1$, with u_k and v_k as in (3.21) and (3.22). Hereafter we set

$$\Omega \equiv \sum_{k=1}^{\infty} \frac{r_{k,\pm}(\tau(z))}{n^k} \,, \tag{4.5}$$

and rely upon the formula

$$\log(1+\Omega) \sim \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\Omega^k}{k} \tag{4.6}$$

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to evaluate the uniform asymptotic expansion

$$\log\left(1+\sum_{k=1}^{\infty}\frac{r_{k,\pm}(\tau(z))}{n^k}\right)\sim\sum_{k=1}^{\infty}\frac{R_{k,\pm}(\tau(z))}{n^k}\,. \tag{4.7}$$

Hence we find [15]

$$R_{1,\pm} = (1\mp\tau)^{-1}\left(\frac{17}{8}\tau\mp\frac{1}{8}\tau^2-\frac{5}{24}\tau^3\pm\frac{5}{24}\tau^4\right), \tag{4.8}$$

$$R_{2,\pm} = (1\mp\tau)^{-2}\left(-\frac{47}{16}\tau^2\pm\frac{15}{8}\tau^3-\frac{21}{16}\tau^4\pm\frac{3}{4}\tau^5-\frac{1}{16}\tau^6\mp\frac{5}{8}\tau^7+\frac{5}{16}\tau^8\right), \tag{4.9}$$

$$\begin{aligned}R_{3,\pm} = (1\mp\tau)^{-3}\Big(&\frac{1721}{384}\tau^3\mp\frac{441}{128}\tau^4+\frac{597}{320}\tau^5\mp\frac{1033}{960}\tau^6+\frac{239}{80}\tau^7\mp\\ &\mp\frac{28}{5}\tau^8+\frac{2431}{576}\tau^9\pm\frac{221}{192}\tau^{10}-\frac{1105}{384}\tau^{11}\pm\frac{1105}{1152}\tau^{12}\Big),\end{aligned} \tag{4.10}$$

and, in general,

$$R_{j,\pm}(\tau(z)) = (1\mp\tau)^{-j}\sum_{a=j}^{4j}C_a^{(j,\pm)}\tau^a\,. \tag{4.11}$$

Note that, at $\tau=1$ (i.e. $z=0$), our $r_{k,+}(\tau)$ and $R_{k,+}(\tau)$ are singular. Such a behaviour is not seen for any of the strongly elliptic boundary-value problems [8].

The $\zeta_B^-(s)$ function is more easily dealt with. It indeed receives contributions from terms equal to

$$\begin{aligned}B_-(s) &\equiv \sum_{n=3}^{\infty}n^{-(2s-2)}\frac{(\sin\pi s)}{\pi}\int_0^{\infty}dz\;z^{-2s}\frac{\partial}{\partial z}\log\left(\frac{\frac{1}{\tau(z)}-\tau(z)}{z^2}\right)\\ &= \omega_0(s)\frac{(\sin\pi s)}{\pi}\int_0^{\infty}dz\;z^{-2s}\frac{\partial}{\partial z}\log\frac{1}{\sqrt{1+z^2}} = -\frac{1}{2}\omega_0(s)\,,\end{aligned} \tag{4.12}$$

and $\sum_{j=1}^{\infty}B_{j,-}(s)$, having defined, with $\lambda=0,j$

$$\omega_\lambda(s)\equiv\sum_{n=3}^{\infty}n^{-(2s+\lambda-2)} = \zeta_H(2s+\lambda-2;3)\,, \tag{4.13}$$

$$B_{j,-}(s)\equiv\omega_j(s)\frac{(\sin\pi s)}{\pi}\int_0^{\infty}dz\;z^{-2s}\frac{\partial}{\partial z}R_{j,-}(\tau(z))\,. \tag{4.14}$$

On using the same method as in [15], such formulae lead to

$$\zeta_B^-(0) = -\frac{5}{4}+\frac{1079}{240}+\frac{5}{2}-\frac{1}{16}\sum_{a=3}^{12}C_a^{(3,-)} = \frac{206}{45}\,, \tag{4.15}$$

a result which agrees with a derivation of $\zeta_B^-(0)$ relying upon the contour method in [24].

To deal with the generalized ζ-function ζ_B^+ we define, in analogy with eq. (4.12),

$$B_+(s)\equiv\omega_0(s)\frac{(\sin\pi s)}{\pi}\int_0^{\infty}dz\;z^{-2s}\frac{\partial}{\partial z}\log\left(\frac{1}{\tau(z)}-\tau(z)\right), \tag{4.16}$$

and, in analogy to eq. (4.14),

$$B_{j,+}(s) \equiv \omega_j(s) \frac{(\sin \pi s)}{\pi} \int_0^\infty dz\, z^{-2s} \frac{\partial}{\partial z} R_{j,+}(\tau(z)) \,. \tag{4.17}$$

Unlike our work in [15] we now exploit eq. (4.4) to evaluate

$$r_{k,+}(\tau) - r_{k,-}(\tau) = \frac{4\tau^2}{(1-\tau^2)} u_{k-1}(\tau) \,, \tag{4.18}$$

and hence we find

$$R_{1,+} = R_{1,-} + \frac{4\tau^2}{(1-\tau^2)} \,, \tag{4.19}$$

$$R_{2,+} = R_{2,-} + \frac{4\tau^2}{(1-\tau^2)} \left(u_1 - \frac{2\tau^2}{(1-\tau^2)} - R_{1,-} \right) , \tag{4.20}$$

$$\begin{aligned} R_{3,+} = R_{3,-} &+ \frac{4\tau^2}{(1-\tau^2)} \left(u_2 - \frac{4\tau^2}{(1-\tau^2)} u_1 - u_1 R_{1,-} - R_{2,-} + \frac{4\tau^2}{(1-\tau^2)} R_{1,-} \right) + \\ &+ \frac{64}{3} \frac{\tau^6}{(1-\tau^2)^3} + \frac{2\tau^2}{(1-\tau^2)} R_{1,-}^2 \,, \end{aligned} \tag{4.21}$$

and so on. This makes it possible to evaluate $B_{j,+}(s) - B_{j,-}(s)$ for all $j = 1, 2, \ldots, \infty$. Only $j = 3$ contributes to $\zeta_B^{\pm}(0)$, because

$$\omega_j(s) \frac{(\sin \pi s)}{\pi} \sim \frac{1}{2} \delta_{j,3} + \tilde{b}_{j,1} s + \mathrm{O}(s^2), \tag{4.22}$$

where [15]

$$\tilde{b}_{j,1} = -1 - 2^{2-j} + \zeta_R(j-2)(1-\delta_{j,3}) + \gamma \delta_{j,3} \,, \tag{4.23}$$

and we find

$$\begin{aligned} B_{3,+}(s) &= B_{3,-}(s) - \\ &- \omega_3(s) \frac{(\sin \pi s)}{\pi} \lim_{\mu \to 1} \int_0^\mu d\tau\, \tau^{2s} (1-\tau)^{-s} (1+\tau)^{-s} \frac{\partial}{\partial \tau} (R_{3,+} - R_{3,-}) \,. \end{aligned} \tag{4.24}$$

Such a limit as μ tends to 1 means that we split the original integration interval according to [15]

$$\int_0^1 d\tau = \int_0^\mu d\tau + \int_\mu^1 d\tau \,.$$

In the first interval on the right-hand side we can integrate at small μ the terms involving negative powers of $(1-\tau)$ in the uniform asymptotics. On the other hand, since the original integral from 0 to 1 is independent of μ, we can, at last, take the $\mu \to 1$ limit [15]. On defining $\widetilde{\mu} \equiv \left(\frac{1}{\mu^2} - 1 \right)^{1/2}$, and writing again x_i for the roots of eq. (1.1), the limit as $\mu \to 1$ of the integral $\int_\mu^1 d\tau$ is reducible to a sum of limits

$$\lim_{\widetilde{\mu} \to 0} \int_0^{\widetilde{\mu}} \frac{z^{1-2s}}{x_i^2 + z^2} dz \,,$$

each of which vanishes since such integrals are equal to

$$\frac{\widetilde{\mu}^{2-2s}{}_2F_1(1,1-s,2-s,-\widetilde{\mu}^2/x_i^2)}{x_i^2(2-2s)}\,.$$

The derivative in the integrand on the right-hand side of eq. (4.24) reads as

$$\frac{\partial}{\partial\tau}(R_{3,+}-R_{3,-})=(1-\tau)^{-4}(1+\tau)^{-4}\left(80\tau^3-24\tau^5+32\tau^7-8\tau^9\right). \tag{4.25}$$

We can thus use the definition

$$Q_\mu(\alpha,\beta,\gamma)\equiv\int_0^\mu \tau^\alpha(1-\tau)^\beta(1+\tau)^\gamma d\tau\,, \tag{4.26}$$

where Q_μ can be evaluated from a hypergeometric function of 2 variables according to [15]

$$Q_\mu(\alpha,\beta,\gamma)=\frac{\mu^{\alpha+1}}{\alpha+1}F_1(\alpha+1,-\beta,-\gamma,\alpha+2;\mu,-\mu)\,, \tag{4.27}$$

to express (4.24) through the functions $Q_\mu(2s+a,-s-4,-s-4)$, with $a=3,5,7,9$. This leads to

$$\begin{aligned}\zeta_B^+(0)&=\zeta_B^-(0)+B_{3,+}(0)-B_{3,-}(0)\\&=\zeta_B^-(0)-\frac{1}{24}\sum_{l=1}^{4}\frac{\Gamma(l+1)}{\Gamma(l-2)}\left[\psi(l+2)-\frac{1}{(l+1)}\right]\kappa_{2l+1}^{(3)}\\&=\frac{206}{45}+2=\frac{296}{45}\,,\end{aligned} \tag{4.28}$$

where $\kappa_{2l+1}^{(3)}$ are the four coefficients of odd powers of τ on the right-hand side of (4.25). Regularity of $\zeta_B^+(s)$ at the origin is guaranteed because $\lim_{s\to 0}s\zeta_B^+(s)$ is proportional to

$$\sum_{l=1}^{4}\frac{\Gamma(l+1)}{\Gamma(l-2)}\kappa_{2l+1}^{(3)}=0\,, \tag{4.29}$$

which is a particular case of the peculiar spectral cancellation (cf. eq. (1.2))

$$\sum_{a=a_{\min}(j)}^{a_{\max}(j)}\frac{\Gamma\left(\frac{(a+1)}{2}\right)}{\Gamma\left(\frac{(a+1)}{2}-j\right)}\kappa_a^{(j)}=0\,, \tag{4.30}$$

where a takes both odd and even values. The case $j=3$ is simpler because then only $\kappa_a^{(j)}$ coefficients with odd a are non-vanishing.

Remaining contributions to $\zeta(0)$, being obtained from strongly elliptic sectors of the boundary-value problem, are easily found to agree with the results in [16], and we find from transverse-traceless, vector, scalar and ghost modes the full $\zeta(0)$ value

$$\zeta(0)=-\frac{278}{45}+\frac{494}{45}-\frac{15}{2}-17+\frac{146}{45}+\frac{757}{90}+\frac{206}{45}+\frac{296}{45}-\frac{149}{45}+\frac{77}{90}+\frac{5}{2}=\frac{142}{45}\,. \tag{4.31}$$

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Since the one-loop prefactor scales as the three-sphere radius raised to a power equal to $\zeta(0)$ [25], we find therefore a vanishing one-loop wave function of the Universe at small three-geometry, which suggests an intriguing quantum avoidance of the cosmological singularity driven by full diffeomorphism invariance. As far as we know, this positive $\zeta(0)$ value for pure gravity is new in the literature. It might have been obtained from our earlier analysis in [15], but, at that time, the cross-check provided by our eqs. (4.18)–(4.30) was missing, and hence we had not yet reached this conclusion. It has been here our aim to focus on pure gravity, but scalar, spinor and gauge-field contributions to the one-loop wave function of the Universe may be included, if necessary [16].

5. Concluding remarks

The idea that the infinite-dimensional invariance group determines completely not only the action functional but also the boundary conditions in (quantum) field theory is very appealing, but such a program faced severe difficulties after the proof in [17] that diffeomorphism-invariant boundary conditions in the de Donder gauge in quantum gravity are incompatible with strong ellipticity.

The more recent work in [15] by the authors changed a lot the overall perspective: although the global heat-kernel asymptotics becomes in general ill-defined, it remains possible to define and evaluate the pure gravity $\zeta(0)$ value at least on the euclidean four-ball, since the associated generalized ζ-function remains regular at the origin. More precisely, the integral representation (4.2) of the generalized ζ-function is legitimate because, for any fixed n, there is a countable infinity of roots x_j and $-x_j$ of eq. (1.1) and they grow approximately linearly with the integer j counting such roots. The function $F_B^{\pm}$ admits therefore a canonical-product representation which ensures that the integral representation (4.2) reproduces the standard definition of generalized ζ-function [15]. Moreover, even though the Mellin transform relating ζ-function to integrated heat kernel cannot be exploited when strong ellipticity is not fulfilled, it remains possible to define a generalized ζ-function. For this purpose, a weaker assumption provides a sufficient condition, i.e. the existence of a sector in the complex plane free of eigenvalues of the leading symbol of the differential operator under consideration [26]. To make sure we have not overlooked some properties of the spectrum, we have been looking for negative eigenvalues or zero-modes, but finding none. Indeed, negative eigenvalues E would imply purely imaginary roots $x = iy$ of eqs. (1.1) and (2.1), but such roots do not exist, as one can check both numerically and analytically; zero-modes would be non-trivial eigenfunctions belonging to zero-eigenvalues, but all modes (tensor, vector, scalar and ghost modes) are combinations of Bessel functions [15] for which this is impossible. As far as we can see, we still find sources of singularities at the origin in the generalized ζ-function resulting from lack of strong ellipticity, but the particular symmetries of the euclidean 4-ball background reduce them to the four terms in eq. (4.29), which add up to zero despite two of them are non-vanishing.

The present paper has provided an important and possibly simpler cross-check of the results in [15]. This is relevant for one-loop quantum cosmology in the Hartle-Hawking program [3], since the wave function of the Universe at small three-geometry [25] becomes

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a wave function about the four-ball background [16]. It is an open question whether it can also be relevant for the AdS/CFT correspondence by virtue of the profound link between AdS/CFT and the Hartle–Hawking wave function of the Universe stressed in [27, section 4.3], the main problem being that AdS/CFT relies upon boundary conditions for metric and other fields at infinity, unlike the case of closed quantum cosmologies.

What is more important, if the Universe had a semiclassical origin [28], our one-loop calculations acquire new interest, further strengthened by the positive $\zeta(0)$ value in eq. (4.31): a positive $\zeta(0)$ value expressing a regular and indeed vanishing one-loop wave function of the Universe in the limit of small three-geometry. We propose to interpret this property as an (unexpected) indication that full diffeomorphism invariance of the boundary-value problem engenders a quantum avoidance of the cosmological singularity. This appears as a conceptually novel perspective, and of course further hard thinking is in order. More precisely, the work by Schleich [25] had found that, on restricting the functional integral to transverse-traceless perturbations, the one-loop semiclassical approximation to the wave function of the Universe diverges at small volumes, at least for the geometry of a three-sphere. The divergence of the wave functional does not necessarily mean that the probability density of the wave functional diverges at small volumes, since the probability density $p[h]$ on the space of wave functionals $\psi[h]$ is given by

$$p[h] = m[h]|\psi|^2[h]\,, \tag{5.1}$$

which includes the effect of the measure $m[h]$ on this space. The scaling of this measure is not known in general. In our manifestly covariant evaluation of the one-loop functional integral for the wave function of the Universe, it would seem untenable to assume that the measure $m[h]$ scales in such a way as to cancel exactly the contribution of $|\psi|^2 \propto (S^3 - \text{radius})^{2\zeta(0)}$. Thus, we conclude that our one-loop wave function of the Universe vanishes at small volume. The normalizability condition of the wave function in the limit of small three-geometry, which is weaker than the condition of its vanishing in this limit, was instead formulated and studied in [29].

Other promising applications of our investigation of the generalized ζ-function deal with the one-loop effective action for braneworld models, as is extensively discussed in [30].

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A. Strong ellipticity

Let A be an elliptic differential operator with leading symbol $A_m(x,\xi)$ and let $\mathcal{K}$ be a cone containing 0 such that, for $\xi \neq 0$, the spectrum of $A_m(x,\xi)$ lies in the complement of $\mathcal{K}$. Let $b^{(0)}(y,\omega,D_r)$ be the leading partial symbol of the boundary operator $\mathcal{B}$. The boundary-value problem $(A,\mathcal{B})$ is then said to be strongly elliptic if, for

$$(0,0) \neq (\omega,\lambda) \in \partial\mathbb{R}^D_+ \times \mathcal{K}\,,$$

the equations [8] (hereafter, r is the geodesic distance to the boundary)

$$A_m(y,0,\omega,D_r)f(r) = \lambda f(r)\,, \tag{A.1}$$

$$\lim_{r\to\infty} f(r) = 0\,, \tag{A.2}$$

$$b^{(0)}(y,\omega,D_r)f(r)[r=0] = g(\omega)\,, \tag{A.3}$$

have a unique solution.

For example, in the case of Dirichlet boundary conditions

$$[\mathcal{B}\phi]_{\partial M} = [\phi]_{\partial M} = 0\,, \tag{A.4}$$

with A an operator of Laplace type, one has the leading symbol $A_2(x,\xi) = |\xi|^2$, and the general equation (A.1) becomes the differential equation

$$A_2(y,0,\omega,D_r)f(r) = \left(-\frac{d^2}{dr^2} + |\omega|^2\right) f(r) = \lambda f(r)\,. \tag{A.5}$$

The general solution of eq. (A.5) reads as

$$f(r) = \alpha(\omega)e^{-r\Lambda} + \beta(\omega)e^{r\Lambda}\,, \tag{A.6}$$

having defined $\Lambda \equiv \sqrt{|\omega|^2 - \lambda}$. The asymptotic condition (A.2) for $r \to \infty$ leads to $\beta = 0$, while eq. (A.3) engenders, by virtue of eq. (A.4), $\alpha(\omega) = g(\omega)$, since the leading partial symbol of the boundary operator reduces to the identity in the Dirichlet case. The boundary conditions (A.4) are therefore strongly elliptic with respect to the cone $\mathbb{C} - \mathbb{R}_+$.

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